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Polynomials of Binomial Type and Approximation Theory

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We study exponential operators $S_\lambda(f, t)$ satisfying for $t \in (A, B)$ with $A > -\infty$, $p(A) = 0$, $p'(A) \neq 0$. We normalize these operators by $A = 0$, $p(A) = 0$, $p'(A) = 1$. We show that there is a one-to-one correspondence between these operators and basic sets of binomial type $\{P_n(x)\}_{n=0}^{\infty}$ with $P_n(x) \geqslant 0$ for $x > 0$. This correspondence is achieved via inverting a family of bilateral Laplace transforms.

$$S_\lambda(f, t) = \int_{-\infty}^{\infty} W(\lambda, t, u) f(u)\, du,$$

$$\frac{\partial W}{\partial t} = \frac{\lambda(u-t)}{p(t)} W, \qquad \lambda > 0,$$

$$\int_{-\infty}^{\infty} W(\lambda, t, u)\, du = 1.$$

1. INTRODUCTION

An exponential operator is a postive linear integral operator

$$S_\lambda(f, t) = \int_{-\infty}^{\infty} W(\lambda, t, u) f(u)\, du, \tag{1.1}$$

whose kernel $W(\lambda, t, u)$, a function or a generalized function, satisfies the partial differential equation

$$\frac{\partial W}{\partial t} = \frac{\lambda(u-t)}{p(t)} W, \qquad \lambda > 0, \tag{1.2}$$

and the normalization condition

$$\int_{-\infty}^{\infty} W(\lambda, t, u)\, du = 1. \tag{1.3}$$

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The function $p(t)$ is defined on a subset of $(-\infty, \infty)$. The domain of t in (1.1) to (1.3) is any component of $\{t \mid p(t) > 0, p \text{ is analytic at } t\}$. The domain of t will be referred to as (A, B). The exponential operators with $p(t)$ a quadratic polynomial were studied by May [2]. Ismail and May [1] studied exponential operators with general $p(t)$. The exponential operators are approximation operators, that is, $S_\lambda(f, t) \to f(t)$ as $\lambda \to \infty$, for certain classes of functions f. The following are examples of exponential operators.

(i) The Gauss–Weierstrass operator; $p(t) = 1$, $t \in (-\infty, \infty)$,

$$G_\lambda(f, t) = (\lambda/2\pi)^{1/2} \int_{-\infty}^{\infty} \exp\{-\lambda(u - t)^2/2\} f(u)\, du.$$

(ii) The Szász operator, $p(t) = t$, $t \in (0, \infty)$,

$$S_\lambda(f, t) = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} f\left(\frac{k}{\lambda}\right).$$

(iii) The Bernstein polynomials, $p(t) = t(1 - t)$, $t \in (0, 1)$,

$$B_n(f, t) = \sum_{k=0}^{n} \binom{n}{k} t^k (1 - t)^{n-k} f\left(\frac{k}{n}\right), \qquad n = 1, 2, \ldots.$$

(iv) The Baskakov operator, $p(t) = t(1 + t)$, $t \in (0, \infty)$,

$$L_\lambda(f, t) = (1 + t)^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda)_k}{k!} \left(\frac{t}{1 + t}\right)^k f\left(\frac{k}{\lambda}\right).$$

(v) The operator R_λ introduced by Ismail an May [1], $p(t) = t(1 + t)^2$, $t \in (0, \infty)$,

$$R_\lambda(f, t) = \exp\{-\lambda t/(1 - t)\} \sum_{k=0}^{\infty} \frac{\lambda(\lambda + k)^{k-1}}{k!} \left(\frac{t}{1 + t}\right)^k \exp\left(\frac{-kt}{1 + t}\right) f\left(\frac{k}{\lambda}\right).$$

References to the above operators and for other exponential operators can be found in [1, 2]. Several approximation properties of these operators can be derived from (1.2) and (1.3). On the other hand it is clear that obtaining explicit forms for these operators is undoubtedly desirable. Ismail and May [1] identified the exponential operator (1.1) as the bilateral (two-sided) Laplace transform

$$S_\lambda(f, t) = \int_{-\infty}^{\infty} \exp\left(-\lambda \int_c^t \frac{\theta - u}{p(\theta)}\, d\theta\right) C(\lambda, u) f(u)\, du \tag{1.4}$$

for some $c \in (A, B)$. This identification led to more insight into the theory of exponential operators; see [1]. The normalization (1.3) becomes

$$\int_{-\infty}^{\infty} \exp\left(-\lambda \int_c^t \frac{\theta - u}{p(\theta)}\, d\theta\right) C(\lambda, u)\, du = 1. \tag{1.5}$$

There is at most one generalized function $C(\lambda, u)$ satisfying (1.5). More precisely, if C_1 and C_2 are generalized functions such that, for some $c \in (A, B)$,

$$\int_{-\infty}^{\infty} \exp\left(-\lambda \int_c^t \frac{\theta - u}{p(\theta)} \, d\theta\right) C_1(\lambda, u) \, du$$
$$= \int_{-\infty}^{\infty} \exp\left(-\lambda \int_c^t \frac{\theta - u}{p(\theta)} \, d\theta\right) C_2(\lambda, u) \, du, \tag{1.6}$$

then $C_1(\lambda, u) = C_2(\lambda, u)$. See Zemanian [6, p. 69].

It is clear that obtaining explicit form for $S_\lambda(f, t)$ depends on inverting (1.5) and that when $A > -\infty$, A will be a singularity or zero of $p(t)$. The present paper treats the inversion of (1.5) when $A > -\infty$ and A is a simple pole of $1/p(t)$. We shall show that in this case the generalized function $C(\lambda, u)$ satisfying (1.5) is a sum of delta functions, say

$$C(\lambda, u) = \sum_{k=0}^{\infty} \phi_k(\lambda) \, \delta(k - \lambda u), \tag{1.7}$$

so that

$$\int_{-\infty}^{\infty} C(\lambda, u) f(u) \, du = \sum_{k=0}^{\infty} \phi_k(\lambda) f\left(\frac{k}{\lambda}\right), \tag{1.8}$$

and the operator $s_\lambda(f, t)$ takes the form

$$S_\lambda(f, t) = \sum_{k=0}^{\infty} \phi_k(\lambda) \exp\left(-\int_c^t \frac{\lambda\theta - k}{p(\theta)} \, d\theta\right) f\left(\frac{k}{\lambda}\right). \tag{1.9}$$

It will turn out that for some $\alpha \neq 0$, the sequence $\{k!\phi_k(\lambda) \, e^{-\alpha\lambda}\}_{k=0}^{\infty}$ is a basic set of binomial type; see Definition 2.1 below. These polynomials of binomial type are well known. The enumeration properties of these polynomials have been studied lately by G.-C. Rota and others in a series of papers on the foundations of combinatorial theory; see [3, 4]. Further references can be bound in [4]. However, the connection between polynomials of binomial type and exponential operators was not knownl We shall show that there is one-to-one correspondence between basic sets of binomial type and the generalized function $C(\lambda, u)$ satisfying (1.5) with $p(t)$ having a simple zero at $t = A$. To illustrate this correspondence we treat, in Section 4, the case $(A, B) = (0, \infty)$, $p(t) = t(1 + 4t)^{1/2}$ which yields a new aproximation operator. We also show that the basic set of binomial type $\{P_n(x)\}_{n=0}^{\infty}$ generated by

$$\sum_{n=0}^{\infty} P_n(x) \frac{\xi^n}{n!} = \exp\{x(\xi + \xi^2)\}$$

gives rise to an exponential operator with $p(t) = ((1 + 8t)^{1/2} - 1/4) \times (1 + 8t)^{1/2}$, which is also a new exponential operator. Section 2 contains basic facts needed in the subsequent analysis. Section 3 contains our main results and exhibits the above mentioned one-to-one correspondence. Finally Section 4 contains some examples.

2. Preliminaries

DEFINITION 2.1 (Sheffer [5]. See also Rota *et al.* [4]). A sequence of polynomials $\{p_n(x)\}_{n=0}^{\infty}$ is called a basic set of binomial type if

(i) $p_n(x)$ is of precise degree n,

(ii) $p_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(y)$,

(iii) $p_0(0) = 1$, $p_n(0) = 0$ for $n = 1, 2, \ldots$.

LEMMA 2.2. (Sheffer [5]. See also Mullin and Rota [3]). *The sequence of polynomials $\{p_n(x)\}_{n=0}^{\infty}$ is a basic of binomial type if and only if*

$$\sum_{n=0}^{\infty} p_n(x) \frac{t^n}{n!} = \exp(xH(t)), \tag{2.1}$$

$H(t)$ is a power series with $H(0) = 0$, $H'(0) \neq 0$.

LEMMA 2.3. *The operator $S_\lambda(f, t)$ of (1.4) is independent of c.*

Proof. Let us denote $C(\lambda, u)$ of (1.5) and $S_\lambda(f, t)$ of (1.4) by $C(\lambda, u, c)$ *and* $S_\lambda(f, t, c)$, respectively. Then by (1.5) we have

$$\int_{-\infty}^{\infty} \exp\left(-\lambda \int_{c_1}^{t} \frac{\theta - u}{p(\theta)}\, d\theta\right) C(\lambda, u, c_1)\, du$$
$$= \int_{-\infty}^{\infty} \exp\left(-\lambda \int_{c_2}^{t} \frac{\theta - u}{p(\theta)}\, d\theta\right) C(\lambda, u, c_2)\, du.$$

Since this is equivalent to

$$\int_{-\infty}^{\infty} \exp\left(-\lambda \int_{c}^{t} \frac{\theta - u}{p(\theta)}\, d\theta\right) \exp\left(+\lambda \int_{c}^{c_1} \frac{\theta - u}{p(\theta)}\, d\theta\right) C(\lambda, u, c_1)\, du$$
$$= \int_{-\infty}^{\infty} \exp\left(-\lambda \int_{c}^{t} \frac{\theta - u}{p(\theta)}\, d\theta\right) \exp\left(\lambda \int_{c}^{c_2} \frac{\theta - u}{p(\theta)}\, d\theta\right) C(\lambda, u, c_2)\, du$$

from (1.6) it follows that

$$\exp\left(\lambda \int_{c}^{c_1} \frac{\theta - u}{p(\theta)}\, d\theta\right) C(\lambda, u, c_1) = \exp\left(\lambda \int_{c}^{c_2} \frac{\theta - u}{p(\theta)}\, d\theta\right) C(\lambda, u, c_2)$$

or

$$C(\lambda, u, c_2) = C(\lambda, u, c_1) \exp \left(-\lambda \int_{c_1}^{c_2} \frac{\theta - u}{p(\theta)}\, d\theta\right). \tag{2.2}$$

The substitution of $C(\lambda, u, c_2)$ from (2.2) in (1.4) yields

$$\begin{aligned} S_\lambda(f, t, c_2) &= \int_{-\infty}^{\infty} \exp\left(-\lambda \int_{c_2}^{t} \frac{\theta - u}{p(\theta)}\, d\theta\right) \\ &\quad \times \exp\left(-\lambda \int_{c_1}^{c_2} \frac{\theta - u}{p(\theta)}\, d\theta\right) C(\lambda, u, c_1) f(u)\, du \\ &= \int_{-\infty}^{\infty} \exp\left(-\lambda \int_{c_1}^{t} \frac{\theta - u}{p(\theta)}\, d\theta\right) C(\lambda, u, c_1) f(u)\, du \\ &= S_\lambda(f, t, c_1). \end{aligned}$$

3. Main Results

Recall that we are studing only the case when $1/P(z)$ has a simple pole at $z = A$. By a linear change of variables in t and u we can take $A = 0$ and

$$1/p(z) = 1/z + h(z) \tag{3.1}$$

with $h(z)$ analytic at $z = 0$.

Lemma 3.1. *If $C(\lambda, u)$ is of the form* (1.7) *then*

$$\exp\{\lambda(\eta(\xi) - \eta(0)) = \sum_{k=0}^{\infty} e^{-\lambda\eta(0)} \phi_k(\lambda)\, \xi^k, \tag{3.2}$$

where

$$\xi = \xi(t) = \frac{t}{c} \exp\left\{\int_c^t h(\theta)\, d\theta\right\} \tag{3.3}$$

and

$$\eta(\xi) = t - c + \int_c^t \theta h(\theta)\, d\theta. \tag{3.4}$$

Furthermore $\eta'(0) \neq 0$ and one can choose c in order to make $\eta(0) \neq 0$.

Proof. The substitution of (1.7) into (1.5) implies

$$\exp\left\{\lambda \int_c^t \frac{\theta\, d\theta}{P(\theta)}\right\} = \sum_{k=0}^{\infty} \phi_k(\lambda) \exp\left\{k \int_c^t \frac{d\theta}{P(\theta)}\right\}; \tag{3.5}$$

hence by (3.1) we get

$$\exp\left\{\lambda\left(t-c+\int_c^t \theta h(\theta)\,d\theta\right)\right\} = \sum_{k=0}^{\infty} \phi_k(\lambda)\left\{\frac{t}{c}\exp\int_c^t h(\theta)\,d\theta\right\}^k.$$

Performing the changes of variables from $(t, h(t))$ to $(\xi, \eta(\xi))$ defined by (3.3) and (3.4) we obtain

$$\exp\{\lambda\eta(\xi)\} = \sum_{k=0}^{\infty} \phi_k(\lambda)\,\xi^k. \tag{3.7}$$

We now choose c in order to make $\eta(0) \neq 0$ and $d\eta/d\xi\,|_{\xi=0} \neq 0$. Observe that $-\eta(0) = c + \int_0^c \theta h(\theta)\,d\theta$ implies that $d\eta(0)/dc = -1$ at $c = 0$. Hence there is an ϵ with $\eta(0) \neq 0$ for all $c \in (0, \epsilon)$. Clearly $d\eta/d\xi\,|_{\xi=0} \neq 0$. Now (3.2) follows immediately from (3.7).

COROLLARY 3.2. *The sequence of functions* $\{e^{-\lambda\eta(0)}k!\,\phi_k(\lambda)\}_{k=1}^{\infty}$ *forms a basic set of binomial type.*

THEOREM 3.3. *If* $A = 0$ *and* $p(z)$ *is of the form* (3.1) *then* $C(\lambda, u)$ *of* (1.5) *is given by*

$$C(\lambda, u) = e^{\lambda\eta(0)} \sum_{k=0}^{\infty} \frac{\psi_k(\lambda)}{k!}\,\delta(k-\lambda u), \tag{3.8}$$

where $\{\psi_k(\lambda)\}_{k=0}^{\infty}$ *is the basic set of binomial type generated by*

$$\sum_{k=0}^{\infty} \psi_k(\lambda)\frac{\xi^k}{k!} = \exp\{\lambda(\eta(\xi)-\eta(0))\}, \tag{3.9}$$

and the functions ξ *and* η *are defined by* (3.3) *and* (3.4).

Proof. The generalized function $C(\lambda, u)$ of (3.8) satisfies (1.5) as can be seen by direct substitution. The theorem then follows from Lemma 2.3.

THEOREM 3.4. *Under the assumptions of Theorem* 3.3 *we have*

$$S_\lambda(f, t) = \exp\{-\lambda(\eta(\xi)-\eta(0))\} \sum_{k=0}^{\infty} \psi_k(\lambda)\frac{\xi^k}{k!} f\left(\frac{k}{\lambda}\right). \tag{3.10}$$

Proof. Substitute (3.8) in (1.4). Note that the operator (3.10) is independent of c or $\eta(0)$.

The operator defined by (3.10) may not be positive. The following is a characterization of such operators.

THEOREM 3.5. *The operator* $S_\lambda(f, t)$ *of* (3.10) *is positive on continuous* *functions with compact support if and only if $\psi_k(\lambda)$ is nonnegative for all k and for $\lambda > 0$.*

Proof. For every k, $k = 0, 1, \ldots$, define the continuous function $f_k(u)$ by

$$\begin{aligned} f_k(u) &= 0 \text{ if } u \in \left(-\infty, \frac{k-1}{\lambda}\right] \cup \left[\frac{k+1}{\lambda}, \infty\right), \\ &= 1 \text{ at } x = k/\lambda, \\ &= \text{linear in } \left[\frac{k-1}{\lambda}, \frac{k}{\lambda}\right] \text{ and } \left[\frac{k}{\lambda}, \frac{k+1}{\lambda}\right]. \end{aligned}$$

Clearly $S_\lambda(f_k, t)$ is a positive multiple of $\psi_k(\lambda)$.

We now prove a converse to Theorem 3.4.

THEOREM 3.6. *Every basic set of polynomials $\{\psi_k(\lambda)\}_{k=0}^\infty$ of binomial type generates an integral operator* (1.1) *whose kernel satisfies* (1.2) *and* (1.3) *with $A = 0$ and $p(t)$ of the form* (3.1). *The integral operator is given explicitely by*

$$S_\lambda(f, t) = \exp\{-\lambda H(\xi)\} \sum_{k=0}^\infty \psi_k(\lambda) \frac{\xi^k}{k!} f\left(\frac{k}{\lambda}\right). \tag{3.11}$$

Proof. By Lemma 2.2 $\{\psi_k(\lambda)\}_{k=0}^\infty$ has the generating function

$$\exp\{\lambda H(\xi)\} = \sum_{k=0}^\infty \psi_k(\lambda) \frac{\xi^k}{k!}. \tag{3.12}$$

Assign $\eta(0)$ arbitrarily and set

$$\eta(\xi) = \eta(0) + H(\xi), \tag{3.13}$$

and

$$t = \xi \frac{d\eta(\xi)}{d\xi} = \xi \frac{dH(\xi)}{d\xi}. \tag{3.14}$$

Define the generalized function $W(\lambda, t, u)$ by

$$W(\lambda, t, u) = \xi^{\lambda u} \exp\{-\lambda(\eta(\xi) - \eta(0))\} \sum_{k=0}^\infty \psi_k(\lambda) \frac{\delta(k - \lambda u)}{k!}. \tag{3.15}$$

Clearly

$$\frac{\partial W}{\partial t} = \frac{\partial W}{\partial \xi}\frac{d\xi}{dt} = \frac{d\xi}{dt}\left(\frac{\lambda u}{\xi} - \lambda \eta'(\xi)\right) W = \frac{\lambda(u - t)}{\xi}\frac{d\xi}{dt} W,$$

which is (1.2) with

$$p(t) = \xi(dt/d\xi). \tag{3.16}$$

It remains to show that $1/p(t)$ has a simple pole at $t = 0$ with residue 1. First it is clear from (3.14) that $t = 0$ when $\xi = 0$. Moreover (3.16) shows that $p(0) = 0$ and that

$$\frac{dp(t)}{dt} = \frac{d}{dt}\left(\xi \frac{dt}{d\xi}\right) = \xi \frac{d}{dt}\left(\frac{dt}{d\xi}\right) + \frac{dt}{d\xi}\frac{d\xi}{dt};$$

hence

$$dp(t)/dt\,|_{t=0} = 1.$$

The analyticity of $p(t)$ is obvious. When $W(\lambda, t, u)$ is given by (3.15) the corresponding integral operator is given by (3.11).

Remark 3.7. The operator (3.11) is independent of the choice of $\eta(0)$. The reason that the value of $\eta(0)$ determines c and we have seen in Lemma 2.4 that the resulting integral operator is independent of c.

THEOREM 3.8. *There is a one-to-one correspondence between basic sets of binomial type* $\{\psi_k(\lambda)\}_{k=0}^{\infty}$, *with* $\psi_k(\lambda) \geqslant 0$ *for all* $\lambda > 0$ *and all* k, *and exponential operators corresponding to* $p(t)$ *with* $p(0) = 0$, $p'(0) = 1$.

Proof. Combine Theorems 3.4 to 3.6 and Remark 3.7.

4. EXAMPLES

We illustrate the theory developed in the previous section by constructing the operators generated by $p(t) = t(1 + 4t)^{1/2}$ and $p(t) = t(1 - t)^{1/2}$. As we shall see the former $p(t)$ generates an exponential while the later $p(t)$ does not, since a member of the corresponding basic set changes sign for $\lambda \in (0, \infty)$. We also compute the function $p(t)$ and the exponential operator associated with $\{P_n(\lambda)\}_{n=0}^{\infty}$

$$\sum_{n=0}^{\infty} P_n(\lambda)\frac{\xi^n}{n!} = \exp\{\lambda(\xi + \xi^2)\}. \tag{4.1}$$

EXAMPLE 1. Consider $p(t) = t(1 + 4t)^{1/2}$, $(A, B) = (0, \infty)$. We take $c = 2$. Hence $h(t) = 1/p(t) - 1/t = 1/t\{(1 + 4t)^{-1/2} - 1\}$. It is easy to see that

$$\int_2^t h(\theta)\, d\theta = \ln\left\{\left(\frac{(1 + 4t)^{1/2} - 1}{t}\right)^2\right\}.$$

The functions $\xi(t)$ and $\eta(\xi)$ of (3.3) and (3.4) are

$$\xi(t) = \frac{t}{2}\left\{\frac{(1+4t)^{1/2}-1}{t}\right\}^2,$$

and

$$\eta(\xi) = \tfrac{1}{2}\{(1+4t)^{1/2}-3\},$$

respectively. Therefore $\eta(0) = -1$ and $(1+4t)^{1/2} = (2+\xi)/(2-\xi)$. Consequently

$$H(\xi) = \xi/(2-\xi),$$

and the corresponding basic set of polynomials, say $\{Q_n(\lambda)\}_{n=0}^{\infty}$, has the generating function

$$\sum_{n=0}^{\infty} \frac{Q_n(\lambda)}{n!}\xi^n = \exp\{\lambda\xi/(2-\xi)\}, \tag{4.2}$$

and it is plain that $Q_n(\lambda) \geqslant 0$ for $\lambda > 0$ and for all k. Equation (4.2) can be written as

$$\sum_{n=0}^{\infty} \frac{2^n Q_n(-\lambda)}{n!}\xi^n = \exp\{-\lambda\xi/(1-\xi),$$

showing that $\{2^n Q_n(-\lambda)\}_{n=0}^{\infty}$ is the basic Laguerre polynomials, see Rota *et al.* [4].

EXAMPLE 2. Take $p(t) = t(1-t)^{1/2}$. Clearly $(A, B) = (0, 1)$. Let $c = \frac{1}{2}$. Straight forward manipulations show that

$$\xi(t) = t\left\{\frac{1-(1-t)^{1/2}}{(2^{1/2}-1)\,t}\right\}^2, \quad \eta(\xi) = 2^{1/2} - 2(1-t)^{1/2}, \quad \eta(0) = 2^{1/2}-2,$$

$$H(\xi) = 2(1-(1-t)^{1/2}), \qquad (1-t)^{1/2} = \frac{1-\xi(3-2(2^{1/2}))}{1+\xi(3-2(2^{1/2}))}.$$

Therefore

$$H(\xi) = \frac{4\xi(3-2(2)^{1/2})}{1+(3-2(2)^{1/2})\,\xi}.$$

Let

$$\sum_{n=0}^{\infty} \frac{q_n(\lambda)}{n!}\xi^n = \exp\{\lambda H(\xi)\} = \exp\left\{\frac{4\lambda\xi(3-2(2)^{1/2})}{1+(3-2(2)^{1/2})\,\xi}\right\}.$$

It is easy to see that

$$q_2(\lambda) = 8(3 - 2(2)^{1/2})^2 \lambda(4\lambda - 1),$$

and hence the resulting S_λ operator is not positive because $q_2(\lambda)$ changes sign.

EXAMPLE 3. Assume that $H(\xi) = \xi(1 + \xi)$ and take $\eta(0) = 1$. From (3.14) we get $t = \xi(1 + 2\xi)$, that is $\xi = \{(1 + 8t)^{1/2} - 1\}/4$. The function $p(t)$ is, by (3.16)

$$p(t) = \xi(1 + 4\xi) = \tfrac{1}{4}((1 + 8t)^{1/2} - 1)(1 + 8t)^{1/2}.$$

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